Table 1 Lift and moment coefficients for pitching and plunging of a typical jet transport wing at a Mach number of 0.8

	Pitching		Plunging	
k	$\bar{C}_z/\alpha_0$	$\bar{C}_m/\alpha_0$	$\bar{C}_z/ikh_o(2/\bar{c})$	$\bar{C}_m/ikh_0(2/\bar{c})$
0.001	$-5.856+i0.0067$	$-0.5653-i0.0023$	$-5.856+i0.0126$	$-0.5653+i0.0009$
0.002	$-5.856+i0.0134$	$-0.5652-i0.0046$	$-5.856+i0.0252$	$-0.5652+i0.0018$
0.004	$-5.855+i0.0267$	$-0.5651-i0.0093$	$-5.855+i0.0505$	$-0.5651+i0.0035$
0.005	$-5.855+i0.0333$	$-0.5650-i0.0116$	$-5.854+i0.0630$	$-0.5650+i0.0044$
0.010	$-5.849+i0.0660$	$-0.5643-i0.0233$	$-5.847+i0.1254$	$-0.5642+i0.0087$
0.020	$-5.826+i0.1278$	$-0.5615+i0.0473$	$-5.820+i0.2459$	$-0.5612+i0.0168$
0.040	$-5.749+i0.2288$	$-0.5524-i0.0981$	$-5.724+i0.4609$	$-0.5514+i0.0297$
0.050	$-5.698+i0.2662$	$-0.5466-i0.1249$	$-5.662+i0.5529$	$-0.5452+i0.0346$
0.100	$-5.398+i0.3251$	$-0.5123-i0.2730$	$-5.278+i0.8558$	$-0.5089+i0.0433$

Table 2 Estimates of $C_{z_\alpha}, C_{m_\alpha}, C_{z_{\dot\alpha}}, C_{m_{\dot\alpha}}, C_{z_q}$ and C_{m_q} based on a single value of k

k	C_{z_α}	C_{m_α}	$C_{z_{\dot\alpha}}$	$C_{m\dot\alpha}$	C_{z_q}	C_{m_q}
0.0	−5.856	−0.5653	–	–	−5.947	−3.206
0.001	−5.856	−0.5653	12.63	0.8883	−5.947	−3.206
0.002	−5.856	−0.5652	12.63	0.8878	−5.946	−3.206
0.004	−5.855	−0.5651	12.62	0.8861	−5.945	−3.206
0.005	−5.855	−0.5650	12.61	0.8848	−5.944	−3.206
0.010	−5.849	−0.5643	12.54	0.8744	−5.936	−3.205
0.020	−5.826	−0.5615	12.30	0.8388	−5.907	−3.203
0.040	−5.749	−0.5524	11.52	0.7432	−5.803	−3.195
0.050	−5.698	−0.5466	11.06	0.6916	−5.734	−3.190
0.100	−5.398	−0.5123	8.56	0.4326	−5.307	−3.162

$$C_{m_{q_{1,3}}} = (1/\bar{c}^3)\ \lfloor x_1 - x \rfloor\ [C_{h_s}]\{(x_3-x)^2\} \tag{17}$$

The moment about the first moment axis for pitching about the second pitch axis is

$$C_{m_{q_{1,4}}} = (1/\bar{c}^3)\ \lfloor x_1 - x \rfloor\ [C_{h_s}]\{(x_4-x)^2\}$$
$$= C_{m_{q_{1,3}}} + 2C_{m_{\alpha_1}}(\Delta x_3/\bar{c}) \tag{18}$$

The moment about the second moment axis for pitching about the first pitch axis is obviously

$$C_{m_{q_{2,3}}} = C_{m_{q_{1,3}}} + C_{z_{q_3}}(\Delta x_1/\bar{c}) \tag{19}$$

Finally, the moment about the second moment axis for pitching about the second pitch axis is

$$C_{m_{q_{2,4}}} = (1/\bar{c}^3)\ \lfloor x_2 - x \rfloor\ [C_{h_s}]\{(x_4-x)^2\}$$
$$= C_{m_{q_{2,3}}} + 2C_{m_{\alpha_2}}(\Delta x_3/\bar{c}) \tag{20}$$

Equations (14), (16), (18), and (20) are variously found in the USAF DATCOM.[7]

Acknowledgment

The authors wish to acknowledge the Northrop Corporation, Aircraft Division, for partial support of this development.

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Aeroelastic Divergence of Unrestrained Vehicles

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[J. Aircraft, **18**, 1072-1073 (1981)]

REFERENCE 1 presented an analysis of an unrestrained vehicle that is in flight and in quasi-static equilibrium by virtue of its inertial forces. A singularity appeared in the formulation that was identified as a physical aeroelastic divergence. This is not the case. Rather than a divergence, it is only a singularity that appears at the dynamic pressure at which the principal axis (i.e., the mean longitudinal reference

Received June 8, 1983.

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axis of Milne[2] when transverse displacement is the primary degree of freedom) remains aligned with the freestream as the vehicle deforms. This becomes apparent by following the alternative derivation of Letsinger[3] to obtain the unrestrained longitudinal stability derivatives referenced to the principal axis.

In the following, the subscript s denotes values at the origin of the structural support system to which the structural influence coefficients (SIC's) are referred. The structural support system defines the attached axes of Milne[2] as tangent, normal, and binormal to the fuselage deflection curve. Consider the symmetrical trim problem of a flexible vehicle. We first consider the restrained vehicle for which the trimmed lift and moment coefficients, referred to some reference area S and chord length $\bar{c}$, are given by

$$\begin{Bmatrix} C_z \\ C_m \end{Bmatrix} = \begin{bmatrix} C_{z_\alpha} & C_{z_\delta} \\ C_{m_\alpha} & C_{m_\delta} \end{bmatrix} \begin{Bmatrix} \alpha_s \\ \delta \end{Bmatrix} + \begin{bmatrix} C_{z_{\ddot z}} & C_{z_{\ddot\theta}} \\ C_{m_{\ddot z}} & C_{m_{\ddot\theta}} \end{bmatrix} \begin{Bmatrix} \ddot z_s/g \\ \ddot\theta_s \bar{c}/2g \end{Bmatrix} \tag{1}$$

in which α_s and δ are the trim angle of attack and control surface incidence, $\ddot z_s$ and $\ddot\theta_s$ are the plunging and pitching accelerations of the principal axis at the longitudinal location of the origin of the support system, and g is the acceleration of gravity.

If we consider the accelerations to be prescribed by the equations of motion, rather than as arbitrary quantities, we have

$$\begin{bmatrix} M & S_s \\ S_s & I_s \end{bmatrix} \begin{Bmatrix} \ddot z_s \\ \ddot\theta_s \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \bar{c} \end{bmatrix} \begin{Bmatrix} C_z \\ C_m \end{Bmatrix} qS \tag{2}$$

Eliminating the accelerations between Eqs. (1) and (2) leads to

$$\begin{Bmatrix} C_z \\ C_m \end{Bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - qS \begin{bmatrix} C_{z_{\ddot z}} & C_{z_{\ddot\theta}} \\ C_{m_{\ddot z}} & C_{m_{\ddot\theta}} \end{bmatrix} \begin{bmatrix} 1/g & 0 \\ 0 & \bar{c}/2g \end{bmatrix} \begin{bmatrix} M & S_s \\ S_s & I_s \end{bmatrix}^{-1} \times \begin{bmatrix} 1 & 0 \\ 0 & \bar{c} \end{bmatrix} \right)^{-1} \begin{bmatrix} C_{z_\alpha} & C_{z_\delta} \\ C_{m_\alpha} & C_{m_\delta} \end{bmatrix} \begin{Bmatrix} \alpha_s \\ \delta \end{Bmatrix} \tag{3a}$$

$$\equiv \begin{bmatrix} C'_{z_\alpha} & C'_{z_\delta} \\ C'_{m_\alpha} & C'_{m_\delta} \end{bmatrix} \begin{Bmatrix} \alpha_s \\ \delta \end{Bmatrix} \tag{3b}$$

The new primed stability derivatives of Eq. (3b) may be regarded as unrestrained stability derivatives, but are still dependent on the selection of the support system for the SIC's.

A set of stability derivatives that is invariant with the SIC support system is found by measuring the angle of attack with respect to the principal axis rather than the structural axis, as shown in Fig. 1. Deflection calculations permit writing

$$\alpha_p = \alpha_s + \beta \tag{4}$$

where

$$\beta = \left(\frac{\partial\beta}{\partial\alpha}\right)\alpha_s + \left(\frac{\partial\beta}{\partial\delta}\right)\delta \tag{5}$$

and the partial derivative coefficients are found from the rotations of the principal axis resulting from the loadings for unit values of α_s and δ. We combine Eqs. (4) and (5) to read

$$\begin{Bmatrix} \alpha_p \\ \delta \end{Bmatrix} = \begin{bmatrix} 1 + \dfrac{\partial\beta}{\partial\alpha} & \dfrac{\partial\beta}{\partial\delta} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_s \\ \delta \end{Bmatrix} \tag{6}$$

or

$$\begin{Bmatrix} \alpha_s \\ \delta \end{Bmatrix} = \frac{1}{1 + \dfrac{\partial\beta}{\partial\alpha}} \begin{bmatrix} 1 & -\dfrac{\partial\beta}{\partial\delta} \\ 0 & 1 + \dfrac{\partial\beta}{\partial\alpha} \end{bmatrix} \begin{Bmatrix} \alpha_p \\ \delta \end{Bmatrix} \tag{7}$$

Substituting Eq. (7) into Eq. (3b) leads to

$$\begin{Bmatrix} C_z \\ C_m \end{Bmatrix} = \frac{1}{1 + \dfrac{\partial\beta}{\partial\alpha}} \begin{bmatrix} C'_{z_\alpha} & C'_{z_\delta} \\ C'_{m_\alpha} & C'_{m_\delta} \end{bmatrix} \times \begin{bmatrix} 1 & -\dfrac{\partial\beta}{\partial\delta} \\ 0 & 1 + \dfrac{\partial\beta}{\partial\alpha} \end{bmatrix} \begin{Bmatrix} \alpha_p \\ \delta \end{Bmatrix} \tag{8a}$$

$$\equiv \begin{bmatrix} C''_{z_\alpha} & C''_{z_\delta} \\ C''_{m_\alpha} & C''_{m_\delta} \end{bmatrix} \begin{Bmatrix} \alpha_p \\ \delta \end{Bmatrix} \tag{8b}$$

The new doubly primed stability derivatives become the unrestrained stability derivatives referred to the principal axis and are invariant with the choice of SIC support system. These are the stability derivatives calculated by FLEXSTAB[4].

In going from Eq. (3b) to Eq. (8b) the critical parameter is the derivative $\partial\beta/\partial\alpha$. Its value depends on the dynamic pressure and, depending on the configuration, can increase or decrease with dynamic pressure (its value at zero dynamic pressure is zero). If it reaches a value of -1.0, the stability derivatives become infinite and reverse their signs at higher dynamic pressures. The variation $\partial\beta/\partial\alpha$ is shown in Table 1 using the example of Ref. 1 with $\lambda = 30$ deg and $r = 0.25$. The value of $qc^4 a_0/GJ$, for which $\partial\beta/\partial\alpha = -1.0$, is 0.992; $\partial\beta/\partial\alpha$ also becomes singular where Eq. (3) becomes infinite (at $qc^4 a_0/GJ = 1.012$ in this example) but this has no practical significance, just as the restrained divergence dynamic pressure ($qc^4 a_0/GJ = 0.887$ in the example) has no physical meaning for the unrestrained vehicle other than to suggest potential dynamic stability problems.

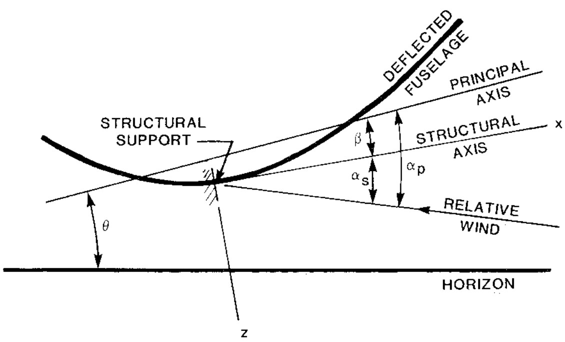

Fig. 1 Geometry of deformed flight vehicle.

Table 1 Variation of $\partial\beta/\partial\alpha$ with dynamic pressure

$qc^4 a_0/GJ$	$\partial\beta/\partial\alpha$
0.0	0.0
0.25	−0.007
0.50	−0.020
0.75	−0.058
0.90	−0.162
0.95	−0.310
0.98	−0.622
0.99	−0.918
0.992	−1.000
1.00	−1.715
1.012	$\mp\infty$
1.02	+2.501

The identity of the dynamic pressures, for which both $\partial\beta/\partial\alpha = -1.0$ and Eq. (18) of Ref. 1 is singular, has not been demonstrated analytically but only numerically. However, numerous numerical experiments have shown that this is always the case, and we conclude that the "divergence" in Ref. 1 is not physical but is only a mathematical property of the principal axis. The limited study of Ref. 5 now suggests that divergence of an unrestrained vehicle should always be investigated by methods of dynamic stability analysis.

Acknowledgement

The author wishes to thank Gary R. Letsinger of the Boeing Aerospace Company for providing him with Ref. 3 which facilitated this correction.

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